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STEADY MULTILAYER FLOWS OF AN IDEAL INCOMPRESSIBLE FLUID OVER AN UNEVEN BOTTOM*

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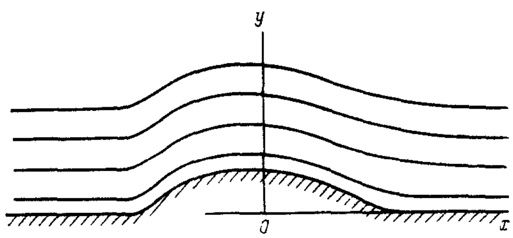
A method of investigating the steady flow of an ideal incompressible stratified fluid over a horizontal bottom with a local irregularity is proposed. The fluid is composed of a finite number of layers, and the density and tangential components of the velocity vector have first-order discontinuities at their boundaries. When mixed Eulerian-Lagrangian variables are used, the initial problem reduces to a boundary value problem with a known boundary, and then to a non-linear integro-differential equation. The solution of the linearized problem is obtained in the form of the sum of Fourier series in terms of the eigenfunctions of the auxiliary spectral problem. The asymptotic behaviour of the waves appearing behind the bottom irregularity is studied together with the simplifications resulting from the assumption that the mean velocity of the unperturbed flow is close to one of the critical velocities of propagation of the long waves. A flow of a two-layer stream with a stepwise density distribution past an obstacle of arbitrary shape is solved as an example.

Two possible formulations concerning the motion of a stratified fluid with jumps in the density and tangential velocity vector component were studied earlier in /1/. The problems of the stability of a stratified flow with density are reviewed in /2/.

1. Formulation of the problem. We consider a steady plane flow of an ideal incompressible stratified fluid over an uneven bottom. The fluid has n layers. The density ρ and tangential component of the velocity vector $\mathbf{V}$ undergo first-order discontinuities at the layer boundaries, and the pressure p is continuous. The Ox axis is directed along the



horizontal level of the bottom, and the Oy axis vertically upwards (see the figure). Let $y = y_k(x)$, $k = 1, 2, \ldots, n$ represent the unknown equations of the boundaries separating the layers, and let $y = y_0(x)$ be a known equation describing the bottom. The function $y_0(x)$ is assumed to be continuous and finite (or decreasing fairly rapidly as $x \to \pm\infty$). It is convenient to assume that when $y > y_n(x)$, we have a fictitious flow for which $\rho = 0$, $\mathbf{V} = 0$, $p = 0$. The functions ρ, $\mathbf{V}$, p are continuously differentiable in every region T_k: $(-\infty < x < +\infty$, $y_{k-1}(x) < y < y_k(x))$, $k = 1, 2, \ldots, n$, and satisfy the system of equations of hydrodynamics which, after the substitution $\mathbf{U} = \sqrt{\rho}\mathbf{V}$, $\mathbf{U} = u\mathbf{i} + v\mathbf{j}$, takes the form

$$(\nabla, \mathbf{U}) = 0, \quad (\mathbf{U}, \nabla\rho) = 0, \quad (\mathbf{U}\nabla)\mathbf{U} = -g\rho\mathbf{j} - \nabla p \tag{1.1}$$

where g is the acceleration due to gravity, and $\mathbf{i}$, $\mathbf{j}$ are unit vectors of the Ox and Oy axes.

Denoting by $\mathbf{N}_k$ the unit normal to the curve $y = y_k(x)$ and by $[F]_k(x)$ the jump in the value of the function $F(x, y)$ on passing through the k-th boundary of separation, we can write the boundary conditions in the form

$$(\mathbf{U}, \mathbf{N}_k) = 0 \text{ when } y = y_k(x), \; k = 1, 2, \ldots, n \tag{1.2}$$
$$[p]_k(x) = 0, \; k = 1, 2, \ldots, n$$

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2. Formulation of the boundary value problem for the stream function.

The first equation of (1.1) enables us to introduce the modified stream function

$$u = \partial\psi/\partial y,\ v = -\partial\psi/\partial x \tag{2.1}$$

If we take into account the fact that the lines $\psi = \text{const}$ coincide with the true stream lines, then the boundary conditions will become

$$\psi(x, y_k(x)) = \psi_k,\ 0 = \psi_0 < \psi_1 < \ldots < \psi_n,\ k = 1, 2, \ldots, n \tag{2.2}$$

where the equations of the separation boundaries $y_k(x)$ and constants ψ_k must be found in the course of the solution.

Assuming that the function $\psi(x, y)$ is continuous, we can show that $\rho = \rho(\psi)$. We assume that the function decreases monotonically, in piecewise smooth, and has first-order discontinuities at the points $\psi_k, k = 1, 2, \ldots, n$. Then (1.1) and (2.1) yield /1–5/

$$\nabla^2\psi + gy\rho'(\psi) = \Phi'(\psi),\ (x, y) \in T_k,\ k = 1, 2, \ldots, n \tag{2.3}$$

$$\Phi(\psi) = {}^1/_2\,(u^2 + v^2) + p + gy\rho(\psi) \tag{2.4}$$

Here and henceforth a prime denotes differentiation with respect to the corresponding argument, and $\Phi(\psi)$ is the Bernoulli function.

From (1.2), (2.2) and (2.4) we obtain the boundary conditions

$$[{}^1/_2\,(\nabla\psi)^2 + gy\rho(\psi) - \Phi(\psi)]_k(x) = 0,\ k = 1, 2, \ldots, n \tag{2.5}$$

$$\psi(x, y_k(x)) = \psi_k,\ [\psi]_k\ (x) = 0,\ \psi(x, y_0(x)) = 0$$

Equation (2.3) and boundary conditions (2.5) contain two unknown, piecewise-smooth functions $\rho(\psi)$ and $\Phi(\psi)$, with first-order discontinuities at the points ψ_k. The form of these functions is determined from the condition than when $x \to -\infty$, a one-dimensional flow is specified with parameters $\rho = R(y), \mathbf{V} = V(y)\,\mathbf{i},\ p = P(y)$ satisfying the hydrodynamic equations and boundary conditions (1.2). The peicewise smooth functions $R(y)$ and $V(y)$ have first-order discontinuities at the points $h_k, k = 1, 2, \ldots, n, 0 < h_1 < \ldots < h_n = H$, and satisfy the conditions

$$R'(y) \leqslant 0,\ R(y) \geqslant R_1 > 0,\ V(y) \geqslant V_1 > 0 \tag{2.6}$$

while the pressure $P(y)$ is hydrostatic

$$P(y) = g\int_y^H R(\xi)\,d\xi$$

Here H is the depth of the unperturbed flow, while R_1 and V_1 are constants. The stream function of one-dimensional flow is found from the solution of the Cauchy problem

$$\psi'(y) = \sqrt{R(y)}\,V(y),\ \psi(0) = 0,\ [\psi]_k(x) = 0,\ k = 1, 2, \ldots, n-1$$

and this in turn yields

$$\psi(y) = \int_0^y a(\xi)\,d\xi,\quad a(y) = \sqrt{R(y)}\,V(y) \tag{2.7}$$

By virtue of the conditions (2.6), Eq.(2.7) has a solution in y

$$y = \eta(\psi),\ 0 \leqslant \psi \leqslant \psi_0,\ \psi_0 = \psi(H) \tag{2.8}$$

where $\eta(\psi)$ is a continuous, monotonically increasing piecewise smooth function and $\eta'(\psi)$ has first-order discontinuities at the points $\psi_k = \psi(h_k),\ k = 1, 2, \ldots, n-1$. Therefore, by virtue of (2.8) we have

$$\rho(\psi) = R(\eta(\psi)),\ V_0(\psi) = V(\eta(\psi)),\ P_0(\psi) = P(\eta(\psi))$$

$$\Phi(\psi) = {}^1/_2\,\rho(\psi)\,V_0^2(\psi) + P_0(\psi) + g\eta(\psi)\,\rho(\psi)$$

Some exact solutions of (2.3) were obtained earlier for the case when the functions $\rho'(\psi)$ and $\Phi'(\psi)$ were specified as linear, and the obstacle was modelled by distributing singularities along the line $x = 0$ [2, 3].

3. Transforming the equation and boundary conditions.

Let us carry out a variable change in Eq.(2.3) and boundary conditions (2.5), taking x and η as the independent variables and $y(x, \eta)$ as the unknown function. The variable η connected with ψ by relation (2.8), plays the part of the Lagrangian coordinate of the fluid particle. We shall also change to dimensionless coordinates, taking the stream depth H as the unit of length, and the numbers

$$R_0 = \frac{1}{H}\int_0^H R(\xi)\,d\xi,\quad c = \frac{1}{HR_0}\int_0^H R(\xi)\,V(\xi)\,d\xi$$

as units of density and velocity, which can be interpreted as the mean density and mean velocity of the one-dimensional flow. As a result we obtain the following boundary value problem for the set of regions G: $(-\infty < x < +\infty,\ \eta_{k-1} < \eta < \eta_k)$, $k = 1, 2, \ldots, n$

$$\frac{\partial}{\partial\eta}\left(a^2(\eta)\frac{1+y_x^2}{2y_\eta^2}\right)-\frac{\partial}{\partial x}\left(a^2(\eta)\frac{y_x}{y_\eta}\right)+\nu R'(\eta)\,y=\Phi_0'(\eta) \tag{3.1}$$

$$\left[a^2(\eta)\frac{1+y_x^2}{2y_\eta^2}+\nu R(\eta)\,y-\Phi_0(\eta)\right]_k(x)=0,\quad k=1,2,\ldots,n \tag{3.2}$$

$$y(x,0)=y_0(x),\ [y]_k(x)=0,\ k=1,2,\ldots,n-1$$

$$\lim_{x\to-\infty} y(x,\eta)=\eta$$

where

$$\Phi_0(\eta)={}^1/_2\,R(\eta)V^2(\eta)+P(\eta)+\nu\eta R(\eta),\quad \nu=gH/c^2 \tag{3.3}$$

The lower indices x and η denote differentiation with respect to the corresponding variable.

The function $y=\eta$ satisfies (3.1) and boundary conditions (3.2), therefore we arrive, after the substitution

$$y=\eta+w(x,\eta) \tag{3.4}$$

at the non-linear boundary value problem for the set G of regions. Here w is the perturbation of the one-dimensional flow, while F_1w and F_2w are non-linear operators

$$\frac{\partial}{\partial\eta}\left(a^2(\eta)\left(\frac{\partial w}{\partial\eta}+F_1w\right)\right)+a^2(\eta)\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}+F_2w\right)-\nu R'(\eta)\,w=0 \tag{3.5}$$

$$\left[a^2(\eta)\left(\frac{\partial w}{\partial\eta}+F_1w\right)-\nu R(\eta)\,w\right]_k(x)=0,\quad k=1,2,\ldots,n$$

$$w(x,0)=y_0(x),\ [w]_k(x)=0,\ k=1,2,\ldots,n-1$$

$$\lim_{x\to-\infty} w(x,\eta)=0$$

$$F_1w=-\frac{2w_\eta^3+3w_\eta^2+w_x^2}{2(1+w_\eta)^2},\quad F_2w=-\frac{w_xw_\eta}{1+w_\eta} \tag{3.6}$$

To obtain the linearized equation of the boundary conditions it is sufficient to put $F_1w=0,\ F_2w=0$ in (3.5).

When carrying out a theoretical study, the passage to the traditional form of the equation containing the Brent-Weisel frequency is not convenient, since the equation loses its divergent form.

4. Basic integro-differential equation. Let us integrate (3.5) over the segment $[\eta,1]$ and use the first boundary condition. Introducing the Lebesque-Stiltjes measure generated by the monotonic function $R(\eta)$, we can write the equation obtained in the form

$$-a^2(\eta)\left(\frac{\partial w}{\partial\eta}+F_1w\right)-\nu\int_\eta^1 w(x,\xi)\,dR(\xi)+\int_\eta^1 a^2(\xi)\left(\frac{\partial^2w}{\partial x^2}(x,\xi)+\frac{\partial}{\partial x}F_2w\right)d\xi=0 \tag{4.1}$$

Let us divide (4.1) by $a^2(\eta)$ and integrate over the segment $[0,\eta]$, using the second and third boundary condition of (3.5). We transform the resulting double integrals, by dividing in the inner integrals the segment of integration $[\tau,1]$ into two subsegments $[\tau,\eta]$ and $[\eta,1]$ and changing the order of integration. After this we introduce a symmetric function of two variables (Green's function)

$$G(\eta,\xi)=\vartheta(\xi-\eta)\int_0^\eta\frac{d\tau}{a^2(\tau)}+\vartheta(\eta-\xi)\int_0^\xi\frac{d\tau}{a^2(\tau)},\quad 0\leqslant\xi,\eta\leqslant1 \tag{4.2}$$

where $\vartheta(t)=0$ when $t<0$, $\vartheta(t)=1$ when $t>0$, $\vartheta(0)={}^1/_2$.

As a result we obtain the following basic integro-differential equation with symmetric and continuous kernels:

$$w(x,\eta)+\nu\int_0^1 G(\eta,\xi)\,w(x,\xi)\,dR(\xi)=y_0(x)- \tag{4.3}$$

$$\int_0^\eta F_1w\,d\xi+\int_0^1 G(\eta,\xi)\,a^2(\xi)\left(\frac{\partial^2w}{\partial x^2}(x,\xi)+\frac{\partial}{\partial x}F_2w\right)d\xi$$

Equation (4.3) describes the formation of the surface and internal waves behind the obstacle. In the case of small perturbations we can restrict ourselves to the linearized equation obtained from (4.3) for $F_1w=0,\ F_2w=0$ and solve it, using the Fourier method of separating the variables. Here it becomes necessary to study certain auxiliary eigenvalue problems with an independent physical meaning.

5. Investigation of the auxiliary integral equation. We begin by investigating the eigenvalue problem for a linear homogeneous integral equation

$$w(\eta)-\nu\int_0^1 G(\eta,\xi)\,w(\xi)\,d\mu(\xi)=0 \tag{5.1}$$

We denote by $L_\mu^2[0, 1]$ a Hilbert space of functions square integrable in the measure $d\mu(\eta) = -dR(\eta)$. The space is of finite dimensions if $R'(\eta) = 0$ with $\eta \in (h_{k-1}, h_k)$, $k = 1, 2, \ldots, n$, and of infinite dimensions if $R'(\eta) < 0$ on the set with non-zero measure. Since the function $G(\eta, \xi)$ is continuous and symmetric, it follows that the integral operator in (5.1) is completely continuous and selfconjugate in $L_\mu^2[0, 1]$, and the Hilbert-Schmidt theory /6–8/ holds for the integral equation (5.1). The basic facts of this theory will be used below without further reference.

All eigenvalues of (5.1) are real. If the space $L_\mu^2[0, 1]$ is of infinite dimension, then a denumerable set of eigenvalues $\{\nu_i\}$ and an orthonormed system of eigenfunctions $\{\varphi_i(\eta)\}$ complete in $L_\mu^2[0, 1]$ both exist. The completeness of the system of eigenfunctions, not only in the domain of values of the integral operator but also in $L_\mu^2[0,1]$, follows from the easily confirmed fact that a continuous function orthogonal to the kernel $G(\eta, \xi)$ must be identically equal to zero. All eigenvalues are isolated and $\nu_i \to +\infty$ as $i \to +\infty$. It can also be shown that all eigenvalues $\nu_i > 0$, since the kernel $G(\eta, \xi)$ is positive definite.

Indeed, the eigenfunction $\varphi(\eta)$ corresponding to the eigenvalue ν must be a solution of the boundary value problem

$$\frac{d}{d\eta}\left(a^2(\eta)\frac{d\varphi}{d\eta}\right)-\nu R'(\eta)\varphi(\eta)=0,\quad h_{k-1}<\eta<h_k,\quad k=1,2,\ldots,n \tag{5.2}$$

$$\left[a^2(\eta)\frac{d\varphi}{d\eta}-\nu R(\eta)\varphi(\eta)\right]_k=0,\quad k=1,2,\ldots,n$$

$$[\varphi]_k=0,\quad k=1,\ldots,n-1,\quad \varphi(0)=0$$

Multiplying (5.2) by $\varphi(\eta)$, integrating from 0 to 1 and repeating the arguments used in deriving (4.1), we obtain

$$\nu=\int_0^1 a^2(\eta)\left(\frac{d\varphi}{d\eta}\right)^2 d\eta\Big/\int_0^1\varphi^2(\eta)\,d\mu(\eta)>0 \tag{5.3}$$

Moreover, all eigenvalues are simple. This follows from the theorem stating that the solution of the Cauchy problem for a second-degree equation is unique. The iterated kernels decompose into bilinear series

$$G^{(m)}(\eta,\xi)=\sum_{i=1}^{\infty}\frac{\varphi_i(\xi)\varphi_i(\eta)}{\nu_i^m}$$

For $m > 2$ the bilinear series converges absolutely and uniformly in ξ and η on $[0, 1]\times[0, 1]$. When $m = 1, 2$, the series converge absolutely and uniformly on any compactum belonging to the set on which the measure $dR(\eta)\,dR(\xi)$ does not vanish (a generalization of Mercer's theorem).

Integrating (5.2) we can obtain a relation of use in subsequent investigation.

$$\nu_i\int_0^1\varphi_i(\xi)\,d\mu(\xi)=a^2(0)\,\varphi_i'(0) \tag{5.4}$$

Considering the inhomogeneous equation (5.1) with right-hand side $f(\eta) \in L_\mu^2[0, 1]$, we find that its solution is expressed in terms of the resolvent $\Gamma(\eta, \xi, \nu)$ as follows:

$$w(\eta)=f(\eta)-\nu\int_0^1\Gamma(\eta,\xi,\nu)\,d\mu(\xi) \tag{5.5}$$

$$\Gamma(\eta,\xi,\nu)=\nu^2\sum_{i=1}^{\infty}\frac{\varphi_i(\xi)\varphi_i(\eta)}{\nu_i^2(\nu-\nu_i)}-\nu G^{(2)}(\eta,\xi)-G(\eta,\xi)$$

In this case the series in (5.5) converges absolutely and uniformly on $[0,1]\times[0,1]$, and the resolvent becomes a continuous function of the variables ξ, η and satisfies the integral relation

$$\Gamma(\eta,\xi,\nu)=\nu\int_0^1 G(\eta,\tau)\,\Gamma(\tau,\xi,\nu)\,d\mu(\tau)-G(\xi,\eta) \tag{5.6}$$

The resolvent is a meromorphic function of the parameter ν with simple poles near the pole $\nu = \nu_l$, and can be expanded in a Laurent series

$$\Gamma(\eta, \xi, \nu) = \frac{\varphi_l(\xi)\varphi_l(\eta)}{\nu - \nu_l} + \sum_{i=0}^{\infty}(\nu - \nu_l)^i M_i(\xi, \eta) \tag{5.7}$$

The coefficients of the series (5.7) are continuous on $[0, 1] \times [0, 1]$.

Let us use formula (5.5) in which the function $f(\eta)$ is equal to the right-hand side of (4.3), and the identity (5.6) for the resolvent. Then we obtain, for the unknown function, $w(x, \eta)$, a non-linear integro-differential equation

$$w(x, \eta) = \chi(\eta, \nu) y_0(x) - \int_0^1 a^2(\xi)\Gamma(\eta, \xi, \nu)\frac{\partial^2 w}{\partial x^2}(x, \xi)\, d\xi + \Phi w \tag{5.8}$$

$$\Phi w = -\int_0^\eta F_1 w\, d\tau + \nu\int_0^1 \Gamma(\eta, \xi, \nu)\left(\int_0^\xi F_1 w\, d\tau\right) d\mu(\xi) -$$

$$\int_0^1 a^2(\xi)\Gamma(\eta, \xi, \nu)\frac{\partial}{\partial x}F_2 w\, d\xi, \quad \chi(\eta, \nu) = 1 - \nu\int_0^1 \Gamma(\eta, \xi, \nu)\, d\mu(\xi)$$

where the non-linear operators $F_1 w$ and $F_2 w$ are given by (3.6).

Thus (5.8) represents an exact integro-differential equation describing wave propagation behind the obstacle. Putting $\Phi w = 0$ in (5.8), we obtain a linear integro-differential equation which we shall solve below using the Fourier method. Here the study of the properties of the eigenvalues and eigenfunctions of the following homogeneous integral Fredholm equation with continuous symmetrizable kernel, is of fundamental importance:

$$z(\eta) = \lambda\int_0^1 a^2(\xi)\Gamma(\eta, \xi, \nu) z(\xi)\, d\xi \tag{5.9}$$

All eigenvalues $\lambda_i(\nu)$ of (5.9) are simple and real. The simplicity of the eigenvalues follows from the equivalence of (5.9) and the eigenvalue boundary value problem for a linear, second-order differential equation and the uniqueness of the solution of the Cauchy problem for such an equation. The question of the sign of $\lambda_i(\nu)$ needs extra investigation. Using the arguments employed in deriving (5.3), we can show that when $\nu < \nu_1$, all $\lambda_i(\nu) < 0$. Thus if the flow rate exceeds any of the critical velocities $c_i = \sqrt{gH/\nu_i}$ of propagation of long waves, then all eigenvalues $\lambda_i < 0$. This means that, as is shown below, at a considerable distance downstream no surface and internal waves are formed.

Since $\Gamma(\eta, \xi, \nu)$ is a meromorphic function of the parameter ν, it follows that the methods of complex analysis can be used to study the dependence of the eigenvalues on the parameter ν. However, a comprehensive study is difficult. In the case when the parameter ν is close to one of the critical values ν_i, the methods of analytic perturbation theory /8/ become effective. It can be shown that a unique eigenvalue $\lambda_l(\nu) \to 0$ exists when $\nu \to \nu_l$, and the asymptotic form of this number and of the corresponding eigenfunction can be found

$$z_l(\eta, \nu) = \varphi_l(\eta)/\sqrt{A_{ll}} + (\nu - \nu_l)\, Q_l(\eta, \nu) \tag{5.10}$$

$$\lambda_l(\nu) = (\nu - \nu_l)/A_{ll} + o(\nu - \nu_l), \quad A_{ll} = \|\varphi_l\|^2 = \int_0^1 a^2(\eta)\varphi_l^2(\eta)\, d\eta$$

Here the function $Q_l(\eta, \nu)$ is analytic near the point $\nu = \nu_l$ and continuous in the variables ξ and η.

A number of useful corollaries can be derived from (5.10). Before anything else, we note that $\lambda_l(\nu) > 0$ when $\nu > \nu_l$ and $\lambda_l(\nu) < 0$ when $\nu < \nu_l$. Since the eigenfunctions $z_i(\eta, \nu)$ are orthogonal with weight $a^2(\eta)$ on the segment $[0, 1]$, it follows that as $\nu \to \nu_l$,

$$(z_i, \varphi_l) = \int_0^1 a^2(\eta) z_i(\eta, \nu)\varphi_l(\eta)\, d\eta = O(\nu - \nu_l), \quad i \neq l \tag{5.11}$$

$$(z_l, \varphi_l) = \sqrt{A_{ll}} + O(\nu - \nu_l), \ \|z_l\| = 1 + O(\nu - \nu_l)$$

We also note that the kernel $\Gamma(\eta, \xi, \nu)$ can be written in the form

$$\Gamma(\eta, \xi, \nu) = z_l(\xi, \nu) z_l(\eta, \nu)/\lambda_l + \Gamma_1(\eta, \xi, \nu) \tag{5.12}$$

where the kernel $\Gamma_1(\eta, \xi, \nu)$ is orthogonal with weight $a^2(\eta)$, and is of the eigenfunction $z_l(\eta, \nu)$. From (5.10) and (5.11) it follows that the kernel $\Gamma_1(\eta, \xi, \nu)$ is analytic in some neighbourhood of the point $\nu = \nu_l$ and, since it is also continuous in η, we have by virtue of Parseval's equality,

$$\sum_{i=1}^{\infty}{}' \frac{z_i^2(\eta, \nu)}{\lambda_i^2} = \int_0^1 \Gamma_1^2(\eta, \xi, \nu)\, d\xi \leqslant C, \quad |\nu - \nu_l| \leqslant \delta, \quad 0 \leqslant \eta \leqslant 1 \tag{5.13}$$

where the index l on the summation sign means that the index $i = l$ is excluded from the summation, and the constants C and δ_0 are independent of η and ν.

6. Solution of the integro-differential equation of the flow past an obstacle. In order to deal with the classical type solutions we will assume, that the function $y_0(x)$ is finite and triply continuously differentiable, and

$$y_0(x) = \delta\Lambda(x), \quad \int_{-\infty}^{+\infty} \Lambda(x)\,dx = 1 \tag{6.1}$$

$$\int_{-\infty}^{+\infty} x\Lambda(x)\,dx = K, \quad \int_{-\infty}^{+\infty} |\Lambda'''(x)|\,dx = M$$

Here δ is a small parameter whose magnitude should, generally speaking, be compatible with ν, and K and M are numbers of the order of unity.

Putting $\Phi w = 0$ in (5.8), we obtain the linearized equation

$$W(x, \eta) = -\int_0^1 a^2(\xi)\Gamma(\eta, \xi, \nu)\frac{\partial^2 W}{\partial x^2}(x, \xi)\,d\xi - \varkappa(\eta)\,y_0''(x) \tag{6.2}$$

$$\varkappa(\eta) = \int_0^1 a^2(\xi)\Gamma(\eta, \xi, \nu)\chi(\xi)\,d\xi$$

$$W(x, \eta) = \omega(x, \eta) - y_0(x)\chi(\eta)$$

We will seek the solution of (6.2) in the form of a Fourier series in the orthonormed system of eigenfunctions of (5.9)

$$W(x, \eta) = \sum_{i=1}^{\infty} B_i(x)\,z_i(\eta, \nu) \tag{6.3}$$

We shall also expand the function $\varkappa(\eta)$ in a Fourier series over the system $\{z_i(\eta, \nu)\}$ to obtain

$$\varkappa(\eta) = \sum_{i=1}^{\infty}\frac{\beta_i}{\lambda_i}z_i(\eta, \nu) \tag{6.4}$$

$$\beta_i = \int_0^1 a^2(\xi)\chi(\xi)z_i(\xi, \nu)\,d\xi = \int_0^1 a^2(\xi)z_i(\xi, \nu)\,d\xi - \frac{\nu}{\lambda_i}\int_0^1 z_i(\xi, \nu)\,d\mu(\xi)$$

If $\nu \to \nu_l + 0$, then by virtue of (5.10) and (5.4) we have

$$\beta_l \approx -\frac{a^2(0)\varphi_l'(0)}{\sqrt{(\nu-\nu_l)\lambda_l}}; \quad \beta_i = O(1), \quad i \neq l \tag{6.5}$$

Substituting the expansions (6.3) and (6.4) into (6.2), we obtain the equations for determining the unknown functions $B_i(x)$

$$B_i''(x) + \lambda_i B_i(x) = -\beta_i y_0''(x), \; i = 1, 2, \ldots \tag{6.6}$$

The solution of (6.6) which vanishes together with its derivative as $x \to -\infty$, has the form

$$B_i(x) = -\frac{\beta_i}{\sqrt{\lambda_i}}\int_{-\infty}^{x} y_0''(\xi)\sin\sqrt{\lambda_i}(x-\xi)\,d\xi, \quad \lambda_i > 0 \tag{6.7}$$

$$B_i(x) = \frac{\beta_i}{2\sqrt{-\lambda_i}}\int_{-\infty}^{+\infty} y_0''(\xi)\exp(-\sqrt{-\lambda_i}\,|x-\xi|)\,d\xi, \quad \lambda_i < 0$$

Integrating by parts we can establish that

$$B_i(x) = \frac{\beta_i}{\lambda_i}(y_0''(x)\operatorname{sgn}\lambda_i - C_i(x)) \tag{6.8}$$

$$C_i(x) = \int_{-\infty}^{x} y_0'''(\xi)\cos\sqrt{\lambda_i}(x-\xi)\,d\xi, \quad \lambda_i > 0$$

$$C_i(x) = \frac{1}{2}\int_{-\infty}^{+\infty} y_0'''(\xi)\exp(-\sqrt{-\lambda_i}\,|x-\xi|)\operatorname{sgn}(x-\xi)\,d\xi, \quad \lambda_i < 0$$

Further, formulas (6.3), (6.8) and the last relation of (6.2) together yield

$$w(x, \eta) = y_0(x)\chi(\eta) - y_0''(x)\varkappa(\eta) + \sum_{i=1}^{\infty}\frac{\beta_i}{\lambda_i}C_i(x)z_i(\eta, \nu) \tag{6.9}$$

Standard estimates based on the use of Bessel's inequality show that the series in (6.9) converges uniformly in x

$$\sum_{i=1}^{\infty}\left|\frac{\beta_i}{\lambda_i}\right||C_i(x)||z_i(\eta,\nu)|\leqslant\frac{\delta M}{2}\sum_{i=1}^{\infty}\left(\beta_i^2+\frac{z_i^2(\eta,\nu)}{\lambda_i^2}\right)\leqslant \tag{6.10}$$
$$\frac{\delta M}{2}\left(\int_0^1 a^2(\xi)\chi^2(\xi)\,d\xi+\max_{0\leqslant\eta\leqslant 1}\int_0^1 a^2(\xi)\Gamma^2(\eta,\xi,\nu)\,d\xi\right)$$

Expansion (6.9) also representes the generalized solution of the integro-differential equation (6.2), provided that the derivatives in (6.9) are interpreted in the sense of the Schwarts theory of distributions. If on the other hand we demand that the function $y_0(x)$ be exceptionally smooth, then the generalized solution will also be classical.

7. Investigation of the wave pattern behind the obstacle. It is evident from formulas (6.8), (6.9) that if the carrier of the function $y_0(x)$ lies on the segment $[-L,L]$, then the flow is unperturbed by the waves at $x<-L$. If on the other hand $x>L$, then we obtain from (6.8) $\lambda_i>0$,

$$C_i(x)=D_i\cos\sqrt{\lambda_i}x+E_i\sin\sqrt{\lambda_i}x \tag{7.1}$$
$$D_i=\int_{-\infty}^{+\infty}y_0'''(\xi)\cos\sqrt{\lambda_i}\,\xi\,d\xi,\quad E_i=\int_{-\infty}^{+\infty}y_0'''(\xi)\sin\sqrt{\lambda_i}\,\xi\,d\xi$$

When $x>L$ and $\lambda_i<0$, we have

$$|C_i(x)|\leqslant\frac{\delta\max|\Lambda'''(x)|}{2\sqrt{|\lambda_i|}}\exp\left(-\sqrt{|\lambda_i|}(x-L)\right) \tag{7.2}$$

Using (7.1) and (6.8) we obtain from (6.9) for $x>L$

$$w(x,\eta)=\sum_{i=1}^{\infty}{}'(D_i\cos\sqrt{\lambda_i}\,x+E_i\sin\sqrt{\lambda_i}\,x)\frac{\beta_i}{\lambda_i}z_i(\eta,\nu)+ \tag{7.3}$$
$$\sum_{i=1}^{\infty}\frac{\beta_i}{\lambda_i}b_i(x)z_i(\eta,\nu),\quad b_i(x)=\begin{cases}0, & \lambda_i>0\\ C_i(x), & \lambda_i<0\end{cases}$$

A prime accompanying the summation sign means that the summation is carried out only over those indices for which $\lambda_i>0$.

Now let $0<\nu-\nu_l<\delta_1$ where the quantity δ_1 is sufficiently small. Using the estimate (5.13) and repeating the estimate (6.10), we obtain

$$\sum_{i=1}^{\infty}{}_l\left|\frac{\beta_i}{\lambda_i}\right||C_i(x)||z_i(\eta,\nu)|\leqslant\frac{\delta M}{2}\Big(\max_{\substack{0\leqslant\nu-\nu_l<\delta_1\\0\leqslant\eta\leqslant 1}}\int_0^1 a^2(\xi)\chi_1^2(\xi)\,d\xi+ \tag{7.4}$$
$$\max_{\substack{0\leqslant\nu-\nu_l<\delta_1\\0\leqslant\eta\leqslant 1}}\int_0^1 a^2(\xi)\Gamma_1^2(\eta,\xi,\nu)\,d\xi\Big)=C\delta$$
$$\chi_1(\eta)=1-\nu\int_0^1\Gamma_1(\eta,\xi,\nu)\,d\mu(\xi)$$

The function $\Gamma_1(\eta,\xi,\nu)$ is defined in (5.12) and the constant C is independent of η and ν. From inequality (7.4) it follows that when $x>L$, then the sum of all terms in (6.9) except the term with index l, is of order δ as $\nu\to\nu_l+0$.

We shall show that when $\lambda_l>0$, then the l-th term is of lower order. Using und (6.8) for $x>L$ and integrating (6.7) by parts, we obtain

$$B_l(x)=-\sqrt{\lambda_l}\,\beta_l\delta\left(\sin\sqrt{\lambda_l}\,x\int_{-\infty}^{x}\Lambda(\xi)\cos\sqrt{\lambda_l}\,\xi\,d\xi-\cos\sqrt{\lambda_l}\,x\int_{-\infty}^{x}\Lambda(\xi)\sin\sqrt{\lambda_l}\,\xi\,d\xi\right) \tag{7.5}$$

where the second term within the brackets is not greater than $\sqrt{\lambda_l}K$.

Substituting (7.5) into (6.9) and using (6.5) and (7.3) we obtain, for $x>L$, $0<\nu-\nu_l<\delta_1$,

$$w(x,\eta)=-\frac{\delta a^2(0)\varphi_l'(0)\varphi_l(\eta)}{\sqrt{(\nu-\nu_l)A_{ll}}}\sin\sqrt{\lambda_l}\,x\int_{-L}^{L}\Lambda(\xi)\cos\sqrt{\lambda_l}\,\xi\,d\xi+\delta w_0(x,\eta,\nu),\quad |w_0(x,\eta,\nu)|\leqslant C \tag{7.6}$$

where the constant C is independent of ν. Further, when $x>L$, we obtain by virtue of (6.1),

$$w(x,\eta)=-\frac{\delta a^2(0)\varphi_l'(0)}{\sqrt{(\nu-\nu_l)A_{ll}}}\sin\sqrt{\lambda_l}\,x\varphi_l(\eta)+O(\delta) \tag{7.7}$$

To satisfy the assumptions of the linear theory, we must make the small parameter δ in

(7.7) compatible with $\nu - \nu_l$. It is sufficient to assume that $\delta = o(\sqrt{\nu - \nu_l})$. It also follows from (6.8) and (6.9) that the perturbation $w(x, \eta)$ is small provided that $y_0(x), y_0'(x), y_0''(x)$ and $y_0'''(x)$. are small. If the assumption that $y_0(x)$ is small is justified, then the assumptions that $y_0'(x), y_0''(x)$ and $y_0'''(x)$ become superfluous. Indeed, we can dismiss these assumptions since the forces of buoyancy have little effect on the character of the flow near the bottom. Here the frictional forces leading to the formation of a boundary layer and stagnation zones near the obstacle are much more important. We can take as $y_0(x)$ the equation of the stream line forming the upper boundary of the boundary layer, the latter being sufficiently smooth.

We also note that formula (7.6) shows the effect of the resonant amplification of the perturbation generated in the stream by the irregularity at the bottom, of order δ. If ν is nearly critical, then the wave amplitude is, generally speaking, of order $\delta/\sqrt{\nu - \nu_l}$.

8. Flow of a two-layer fluid past an obstacle.

We consider, as the simplest example illustrating the proposed method, the flow of a two-layer fluid with stepwise density distribution, past an obstacle of arbitrary shape ($\varepsilon > 0$ is a small parameter)

$$R(\eta) = \begin{cases} 1 + \varepsilon h_0, & 0 \leqslant \eta < h \\ 1 - \varepsilon h, & h \leqslant \eta < 1, \\ 0, & \eta > 1 \end{cases} \qquad V(\eta) = \begin{cases} 1, & 0 \leqslant \eta < 1 \\ 0, & \eta \geqslant 1 \end{cases} \qquad \int_0^1 R(\eta)\, d\eta = 1, \quad h + h_0 = 1$$

Then the function

$$q(\eta) = \int_0^\eta \frac{d\tau}{R(\tau)}$$

will be peicewise linear. We have, with accuracy of order ε^2

$$q(0) = 0, \; q(h) = h - \varepsilon h h_0 + \varepsilon^2 h h_0^2, \; q(1) = 1 + \varepsilon^2 h h_0$$

Green's function $G(\eta, \xi)$ is also piecewise linear and

$$G(h, h) = G(h, 1) = q(h), \; G(1, 1) = q(1)$$

From the integral equation (5.1) we have

$$\nu^{-1}\varphi(\eta) = \varepsilon A G(\eta, h) + (1 - \varepsilon h) B G(\eta, 1), \; A = \varphi(h), B = \varphi(1) \tag{8.1}$$

Substituting into (8.1) $\eta = h$ and $\eta = 1$, we arrive at the algebraic system in the unknowns A and B. Equating to zero the determinant of this system, we obtain the characteristic equation, from which we obtain

$$\nu_1^{-1} = 1 - \varepsilon h h_0 + \varepsilon^2 h h_0^3, \; \nu_2^{-1} = \varepsilon h h_0 - \varepsilon^2 h h_0^3$$

The corresponding eigenfunctions are piecewise linear and

$$\varphi_1(h) = q(h), \; \varphi_1(1) = 1 - \varepsilon h h_0 + \varepsilon^2 h h_0 (1 + h_0^2)$$
$$\varphi_2(h) = q(h), \; \varphi_2(1) = -\varepsilon h^2 + \varepsilon^2 h^2 h_0 (1 + h_0)$$

The normed eigenfunctions have, with an accuracy of up to $O(\varepsilon)$, the form

$$\varphi_1(\eta) = \eta, \; 0 \leqslant \eta \leqslant 1, \quad \varphi_2(\eta) = \frac{1}{\sqrt{\varepsilon}\, h h_0} \begin{cases} h_0 \eta, & 0 \leqslant \eta < h \\ h(1 - \eta), & h \leqslant \eta \leqslant 1 \end{cases} \tag{8.2}$$

We see that the function $\varphi_1(\eta)$ attains its maximum when $\eta = 1$, and $\varphi_2(\eta)$ when $\eta = h$. The quantities appearing in (7.7) have the following form with an accuracy of up to $O(\varepsilon)$:

$$\varphi_0'(0) = 1, \quad \varphi_2'(0) = \frac{1}{\sqrt{\varepsilon}\, h}, \quad A_{11} = \frac{1}{3}, \quad A_{22} = \frac{1}{3\varepsilon}$$

$$a(0) = 1, \quad \lambda_1 = 3(\nu - 1), \quad \lambda_2 = \frac{3}{h h_0}(\varepsilon \nu h h_0 - 1)$$

From (8.2) it follows that in the case of the first (barotropic) mode the greatest perturbation occurs at the free boundary. We find the asymptotic expression for the free boundary from (3.4) and (7.7)

$$y_1(x) \approx 1 - \delta \sqrt{\frac{3}{\nu - 1}} \sin \sqrt{3(\nu - 1)}\, x$$

For the second (baroclinic) mode we find the greatest perturbation occurs at the boundary of separation whose asymptotic formula has the form

$$y_2(x) \approx h\left(1 - \frac{\delta}{h}\sqrt{\frac{3h_0}{(\nu^\circ - 1)h}} \sin \sqrt{\frac{3h(\nu^\circ - 1)}{h_0}}\, \frac{x}{h}\right) \tag{8.3}$$
$$\nu^* = \varepsilon \nu h h_0 = \frac{g H_1 H_2 (R_1 - R_2)}{c^2 (R_1 H_1 + R_2 H_2)}$$

The quantitiy ν° on the right hand side of (8.3) is written in terms of the dimensional variables where R_1, H_1, and' R_2, H_2 denote the density and thickness of the lower and upper layer respectively. The critical value of the parameter ν^* is equal to unity, and the critical value of the rate of propagation of the long waves at the boundary of separation corresponds to it.

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ON VORTICITY-INDUCED WAVES IN A HOMOGENEOUS INCOMPRESSIBLE FLUID*

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The existence of vortex-induced waves in a homogeneous incompressible fluid is proved. The boundary of the vortex represents a cylindrical rotating fluid surface of stable form. The non-linear dispersion relation, the form of the vortex and the stream function are found for the vortices bounded by an almost circular cylinder, and for the vortex-induced waves. The character and special features of the oscillation of the velocity field are explained.

The problem is reduced to that of proving the existence of a branching solution of the non-linear integral equation and to effective determination of the solution and the bifurcation value of the parameter. An iterative method is proposed enabling the simultaneous determination at every stage of the approximation to the branching solution and bifurcation value of the parameter. The convergence of the method over a certain range of parameters is proved.

The possibility of the existence of rotating cylindrical vortices retaining the non-circular form of the transverse cross-section was shown by Lamb /1/ who obtained the linearized dispersion relation (3.2). Following /2/ we shall call such vortices "vortons". Deem and Zabusky carried out a numerical experiment in /2/ and they suggest that the result proves the existence of vortons. It was also found that the rotation frequency of these vortices is less than the value obtained from (3.2).

The vortons and vorton-induced waves are of interest (see the foreword to /2/), since the results of the numerical experiment are interpreted as manifestations of the "soliton-like" behaviour of the waves in a two-dimensional medium.

1. Formulation of the problem. Consider the flow of an ideal homogeneous incompressible and unbounded fluid in a direction parallel to the xoy plane (Fig.1). We denote by ox, oy the fixed axes and by ox_1, oy_1 the axes rotating with constant angular velocity Ω, r, θ are polar coordinates in the xoy plane, r, β are polar coordinates in the x_1oy_1 plane, t is time, $\mathbf{q}(r, \theta, t)$ denotes the absolute velocity of the fluid (relative to the fixed axes), q_r, $q_\theta \equiv q_\beta$ is the radial and tangential component of the absolute velocity, and $\zeta = \operatorname{rot} \mathbf{q} = \zeta \mathbf{i}_z$, $\mathbf{i}_z$ is the unit vector normal to the xoy plane (and to the x_1oy_1) plane). When $t = 0$, the ox and ox_1 axes coincide.

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